Journal of Algebra, Number Theory: Advances and Applications
Volume 5, Number 1, 2011, Pages 21-40

CHARACTERIZATIONS OF LIE DERIVATIONS OF GENERALIZED MATRIX ALGEBRAS

JIANBIN GUO

Department of Mathematics
East China University of Science and Technology
Shanghai 200237
P. R. China
e-mail: guojianbin19840410@yahoo.com

Abstract

We say that L is a Lie derivable mapping at G from an algebra $\mathcal{A}$ into itself, if $L([A, B]) = [L(A), B] + [A, L(B)]$ for any $A, B \in \mathcal{A}$ with $AB = G$. Let $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$ be a generalized matrix algebra. In this paper, we prove that if L is a Lie derivable mapping at 0 (resp., $I_{\mathcal{A}} \oplus 0$) from $\mathcal{U}$ into itself, then L can be expressed as the sum of a derivation of $\mathcal{U}$ and a linear mapping with image in the center vanishing at commutators $[S, T]$, where $S, T \in \mathcal{U}$ with $ST = 0$ (resp., $ST = I_{\mathcal{A}} \oplus 0$).

1. Introduction

Let $\mathcal{R}$ be a unital ring and $\mathcal{A}$ be a unital $\mathcal{R}$-algebra. A linear mapping δ from $\mathcal{A}$ into itself is called a *derivation*, if $\delta(xy) = \delta(x)y + x\delta(y)$ for any $x, y \in \mathcal{A}$. A linear mapping L from $\mathcal{A}$ into itself is called a *Lie derivation*, if $L([x, y]) = [L(x), y] + [x, L(y)]$ for any $x, y \in \mathcal{A}$, where

2010 Mathematics Subject Classification: 16W25, 47B47, 47L35.

Keywords and phrases: derivation, generalized matrix algebra, Lie derivation.

Received March 30, 2011

$[x, y] = xy - yx$ is the usual Lie product. The questions of characterizing Lie derivations and revealing the relationship between Lie derivations and derivations have attracted some authors' attention. In [9], Mathieu and Villena proved that every linear Lie derivation on a C^*-algebra can be decomposed into the sum of a derivation and a center-valued trace. For other results, see [1, 3, 4, 6, 7, 10] and the references therein.

Recently, there have been a number of papers on the study of conditions under which derivations on algebras can be completely determined by their action on some subsets of elements (for example, see [2, 5, 12] and the references therein). We say that L is a *Lie derivable mapping* at G from $\mathcal{A}$ into itself, if $L([A, B]) = [L(A), B] + [A, L(B)]$ for any $A, B \in \mathcal{A}$ with $AB = G$. In [8], Lu and Jing first studied the local actions of Lie derivations and showed that if L is a Lie derivable mapping at 0 (resp., a fixed nontrivial idempotent P) from $B(X)$ into itself, then L can be expressed as the sum of a derivation of $B(X)$ and a linear mapping with image in the center vanishing at commutators $[A, B]$, where $A, B \in B(X)$ with $AB = 0$ (resp., $AB = P$).

Let $\mathcal{R}$ be a unital ring. A *Morita context* is a set $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N})$ and two mappings ϕ and φ, where $\mathcal{A}$ and $\mathcal{B}$ are two $\mathcal{R}$-algebras, $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule, and $\mathcal{N}$ is a $(\mathcal{B}, \mathcal{A})$-bimodule. The mappings $\phi : \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A}$ and $\varphi : \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \to \mathcal{B}$ are two bimodule homomorphisms satisfying the following associativity: $\phi(M \otimes N)M' = M\varphi(N \otimes M')$ and $\varphi(N \otimes M)N' = N\phi(M \otimes N')$, for all $M, M' \in \mathcal{M}$ and $N, N' \in \mathcal{N}$. These conditions insure that the set

$$\begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} A & M \\ N & B \end{bmatrix} \middle| A \in \mathcal{A}, M \in \mathcal{M}, N \in \mathcal{N}, B \in \mathcal{B} \right\},$$

form an $\mathcal{R}$-algebra under usual matrix operations. We call such an $\mathcal{R}$-algebra a *generalized matrix algebra* and denoted by $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$, where $\mathcal{A}$ and $\mathcal{B}$ are two unital algebras and at least one of the two bimodules $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. This kind of algebra was first introduced by Sands in [11]. Obviously, when $\mathcal{M} = 0$ or $\mathcal{N} = 0$, $\mathcal{U}$ degenerates to the triangular algebra. We denote $I_{\mathcal{A}}$ the unit element in $\mathcal{A}$, $I_{\mathcal{B}}$ the unit element in $\mathcal{B}$, and $A \oplus B$ the element $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ in $\mathcal{U}$. The purpose of this paper is to consider the local actions of Lie derivations of generalized matrix algebras. We prove that if L is a Lie derivable mapping at 0 (resp., $I_{\mathcal{A}} \oplus 0$) from $\mathcal{U}$ into itself, then L can be expressed as the sum of a derivation of $\mathcal{U}$ and a linear mapping with image in the center vanishing at commutators $[S, T]$, where $S, T \in \mathcal{U}$ with $ST = 0$ (resp., $ST = I_{\mathcal{A}} \oplus 0$).

In this paper, the center of an algebra $\mathcal{A}$ is denoted by $Z(\mathcal{A})$. Given a generalized matrix algebra $\mathcal{U} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$, we define two natural projections $\pi_{\mathcal{A}} : \mathcal{U} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{U} \to \mathcal{B}$ by

$$\pi_{\mathcal{A}}\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right) = A \quad \text{and} \quad \pi_{\mathcal{B}}\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right) = B.$$

Given an integer $n \geq 2$, we say that the *characteristic* of an algebra $\mathcal{A}$ is not n, if for every $A \in \mathcal{A}$, $nA = 0$ implies $A = 0$. In this paper, we always assume that the characteristic of $\mathcal{U}$ is not 2.

2. Lie Derivable Mapping at Zero-Product Elements

We call a left (resp., right) $\mathcal{A}$-module $\mathcal{M}$ is *faithful*, if for any $A \in \mathcal{A}$ and $M \in \mathcal{M}$, $AM = 0$ (resp., $MA = 0$) implies $A = 0$. In this section, we study the Lie derivable mapping at zero-product elements. Our main result is the following:

Theorem 2.1. *Let* $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$ *be a generalized matrix algebra and L be a Lie derivable mapping at* 0 *from* $\mathcal{U}$ *into itself. Suppose that* $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{U}))$, $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{U}))$, *and one of the following conditions holds*:

(1) *M is a faithful left* $\mathcal{A}$*-module and a faithful right* $\mathcal{B}$*-module*;

(2) *M is a faithful left* $\mathcal{A}$*-module and N is a faithful left* $\mathcal{B}$*-module*;

(3) *N is a faithful right* $\mathcal{A}$*-module and M is a faithful right* $\mathcal{B}$*-module*;

(4) *N is a faithful right* $\mathcal{A}$*-module and a faithful left* $\mathcal{B}$*-module.*

Then L can be expressed as $\delta + h$, *where* δ *is a derivation on* $\mathcal{U}$, *and* $h : \mathcal{U} \to Z(\mathcal{U})$ *is a linear mapping, vanishing at commutators* $[S, T]$, *where* $S, T \in \mathcal{U}$ *with* $ST = 0$.

Since L is linear, for any $A \in \mathcal{A}$, $M \in \mathcal{M}$, $N \in \mathcal{N}$, and $B \in \mathcal{B}$, we may write

$$L\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right)$$

$$= \begin{bmatrix} a_{11}(A) + b_{11}(B) + c_{11}(M) + d_{11}(N) & a_{12}(A) + b_{12}(B) + c_{12}(M) + d_{12}(N) \\ a_{21}(A) + b_{21}(B) + c_{21}(M) + d_{21}(N) & a_{22}(A) + b_{22}(B) + c_{22}(M) + d_{22}(N) \end{bmatrix},$$

where a_{ij}, b_{ij}, c_{ij}, and d_{ij} are linear mappings, $i, j \in \{1, 2\}$.

To prove Theorem 2.1, we first show a lemma and two propositions.

Lemma 2.2. *Let* $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$ *be a generalized matrix algebra and L be a Lie derivable mapping at* 0 *from* $\mathcal{U}$ *into itself. Then*

$$L\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right)$$

$$= \begin{bmatrix} a_{11}(A) - MN_0 - M_0N + b_{11}(B) & AM_0 + c_{12}(M) - M_0B \\ N_0A + d_{21}(N) - BN_0 & a_{22}(A) + N_0M + NM_0 + b_{22}(B) \end{bmatrix},$$

where $M_0 \in M$, $N_0 \in N$, *and* $a_{11} : \mathcal{A} \to \mathcal{A}$, $b_{22} : \mathcal{B} \to \mathcal{B}$, $a_{22} : \mathcal{A} \to Z(\mathcal{B})$, $b_{11} : \mathcal{B} \to Z(\mathcal{A})$ *are all linear mappings satisfying*

$$c_{12}(AM) = Ac_{12}(M) + a_{11}(A)M - Ma_{22}(A),$$

$$c_{12}(MB) = c_{12}(M)B + Mb_{22}(B) - b_{11}(B)M,$$

$$d_{21}(NA) = d_{21}(N)A + Na_{11}(A) - a_{22}(A)N,$$

$$d_{21}(BN) = Bd_{21}(N) + b_{22}(B)N - Nb_{11}(B),$$

$$a_{11}(MN) = c_{12}(M)N + Md_{21}(N) + b_{11}(NM),$$

$$b_{22}(NM) = Nc_{12}(M) + d_{21}(N)M + a_{22}(MN).$$

Proof. We prove the lemma by two steps:

Step 1. For any $A \in \mathcal{A}$, M_1, $M_2 \in \mathcal{M}$, $B \in \mathcal{B}$, let $S = \begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix}$. Then $TS = 0$ and

$$\begin{bmatrix} c_{11}(AM_2 + M_1B) & c_{12}(AM_2 + M_1B) \\ c_{21}(AM_2 + M_1B) & c_{22}(AM_2 + M_1B) \end{bmatrix}$$

$$= L([S, T]) = [L(S), T] + [S, L(T)]$$

$$= \begin{bmatrix} a_{11}(A) + c_{11}(M_1) & a_{12}(A) + c_{12}(M_1) \\ a_{21}(A) + c_{21}(M_1) & a_{22}(A) + c_{22}(M_1) \end{bmatrix} \begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix}$$

$$- \begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix} \begin{bmatrix} a_{11}(A) + c_{11}(M_1) & a_{12}(A) + c_{12}(M_1) \\ a_{21}(A) + c_{21}(M_1) & a_{22}(A) + c_{22}(M_1) \end{bmatrix}$$

$$+ \begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11}(M_2) + b_{11}(B) & c_{12}(M_2) + b_{12}(B) \\ c_{21}(M_2) + b_{21}(B) & c_{22}(M_2) + b_{22}(B) \end{bmatrix}$$

$$- \begin{bmatrix} c_{11}(M_2) + b_{11}(B) & c_{12}(M_2) + b_{12}(B) \\ c_{21}(M_2) + b_{21}(B) & c_{22}(M_2) + b_{22}(B) \end{bmatrix} \begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix}.$$

The above matrix equation implies the following four equations:

$$c_{11}(AM_2 + M_1B) = Ac_{11}(M_2) + Ab_{11}(B) + M_1c_{21}(M_2) + M_1b_{21}(B) - M_2a_{21}(A) - M_2c_{21}(M_1) - c_{11}(M_2)A - b_{11}(B)A, \tag{2.1}$$

$$c_{12}(AM_2 + M_1B) = a_{11}(A)M_2 + c_{11}(M_1)M_2 + a_{12}(A)B + c_{12}(M_1)B + Ac_{12}(M_2) + Ab_{12}(B) + M_1c_{22}(M_2) + M_1b_{22}(B) - M_2a_{22}(A) - M_2c_{22}(M_1) - c_{11}(M_2)M_1 - b_{11}(B)M_1, \tag{2.2}$$

$$c_{21}(AM_2 + M_1B) = -Ba_{21}(A) - Bc_{21}(M_1) - c_{21}(M_2)A - b_{21}(B)A, \tag{2.3}$$

$$c_{22}(AM_2 + M_1B) = a_{21}(A)M_2 + c_{21}(M_1)M_2 + a_{22}(A)B + c_{22}(M_1)B - Ba_{22}(A) - Bc_{22}(M_1) - c_{21}(M_2)M_1 - b_{21}(B)M_1. \tag{2.4}$$

Taking $M_1 = M_2 = 0$ in (2.1)-(2.4), we have

$$[A, b_{11}(B)] = 0, \quad [a_{22}(A), B] = 0, \tag{2.5}$$

$$a_{12}(A)B = -Ab_{12}(B), \tag{2.6}$$

$$Ba_{21}(A) = -b_{21}(B)A, \tag{2.7}$$

for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. By (2.5), we have

$$b_{11}(B) \in Z(\mathcal{A}) \quad \text{and} \quad a_{22}(A) \in Z(\mathcal{B}), \tag{2.8}$$

for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. Taking $A = I_{\mathcal{A}}$ and $B = I_{\mathcal{B}}$ in (2.6), we have $a_{12}(I_{\mathcal{A}}) = -b_{12}(I_{\mathcal{B}})$. Let $M_0 = a_{12}(I_{\mathcal{A}})$. Then taking $A = I_{\mathcal{A}}$ and $B = I_{\mathcal{B}}$ in (2.6), respectively, we obtain

$$a_{12}(A) = AM_0 \quad \text{and} \quad b_{12}(B) = -M_0B, \tag{2.9}$$

for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. Similarly, let $a_{21}(I_{\mathcal{A}}) = N_0$, by (2.7), we obtain

$$a_{21}(A) = N_0 A \quad \text{and} \quad b_{21}(B) = -BN_0, \tag{2.10}$$

for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$.

Taking $B = 0$ and $M_1 = 0$ in (2.2)-(2.4), we have

$$c_{12}(AM) = Ac_{12}(M) + a_{11}(A)M - Ma_{22}(A), \tag{2.11}$$

$$c_{21}(AM) = -c_{21}(M)A, \tag{2.12}$$

$$c_{22}(AM) = a_{21}(A)M, \tag{2.13}$$

for every $A \in \mathcal{A}$ and every $M \in \mathcal{M}$. Since $\mathrm{char}(\mathcal{U}) \neq 2$, by taking $A = I_{\mathcal{A}}$ in (2.12) and (2.13), we obtain

$$c_{21}(M) = 0 \quad \text{and} \quad c_{22}(M) = N_0 M, \tag{2.14}$$

for every $M \in \mathcal{M}$. Similarly, by (2.1) and (2.2), we obtain

$$c_{12}(MB) = c_{12}(M)B + Mb_{22}(B) - b_{11}(B)M, \tag{2.15}$$

$$c_{11}(M) = -MN_0, \tag{2.16}$$

for every $B \in \mathcal{B}$ and every $M \in \mathcal{M}$.

Symmetrically, by considering $S = \begin{bmatrix} 0 & 0 \\ N_1 & B \end{bmatrix}$ and $T = \begin{bmatrix} A & 0 \\ N2 & 0 \end{bmatrix}$, we arrive at

$$d_{11}(N) = -M_0 N, \quad d_{12}(N) = 0, \quad d_{22}(N) = NM_0, \tag{2.17}$$

$$d_{21}(NA) = d_{21}(N)A + Na_{11}(A) - a_{22}(A)N, \tag{2.18}$$

$$d_{21}(BN) = Bd_{21}(N) + b_{22}(B)N - Nb_{11}(B), \tag{2.19}$$

for every $A \in \mathcal{A}$, every $B \in \mathcal{B}$, and every $N \in \mathcal{N}$.

Step 2. For any $M \in \mathcal{M}$ and $N \in \mathcal{N}$, let $S = \begin{bmatrix} MN & -M \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ N & 0 \end{bmatrix}$. Then $ST = 0$ and

$$-\begin{bmatrix} a_{11}(MN) + MN_0 - M_0NMN - b_{11}(NM) & MNM_0 - c_{12}(M) + M_0NM \\ N_0MN + d_{21}(NMN) + NMN_0 & a_{22}(MN) - N_0M + NMNM_0 - b_{22}(NM) \end{bmatrix}$$

$$= L([S, T]) = [L(S), T] + [S, L(T)]$$

$$= \begin{bmatrix} a_{11}(MN) + MN_0 & MNM_0 - c_{12}(M) \\ N_0MN & a_{22}(MN) - N_0M \end{bmatrix} \begin{bmatrix} I_{\mathcal{A}} & 0 \\ N & 0 \end{bmatrix}$$

$$- \begin{bmatrix} I_{\mathcal{A}} & 0 \\ N & 0 \end{bmatrix} \begin{bmatrix} a_{11}(MN) + MN_0 & MNM_0 - c_{12}(M) \\ N_0MN & a_{22}(MN) - N_0M \end{bmatrix}$$

$$+ \begin{bmatrix} MN & -M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11}(I_{\mathcal{A}}) - M_0N & M_0 \\ N_0 + d_{21}(N) & a_{22}(I_{\mathcal{A}}) + NM_0 \end{bmatrix}$$

$$- \begin{bmatrix} a_{11}(I_{\mathcal{A}}) - M_0N & M_0 \\ N_0 + d_{21}(N) & a_{22}(I_{\mathcal{A}}) + NM_0 \end{bmatrix} \begin{bmatrix} MN & -M \\ 0 & 0 \end{bmatrix}.$$

The above matrix equation implies

$$a_{11}(MN) = c_{12}(M)N + Md_{21}(N) + b_{11}(NM) + a_{11}(I_{\mathcal{A}})MN - MNa_{11}(I_{\mathcal{A}}), \tag{2.20}$$

$$b_{22}(NM) = Nc_{12}(M) + d_{21}(N)M + a_{22}(MN). \tag{2.21}$$

By (2.11) and (2.18), $c_{12}(AM) = Ac_{12}(M) + a_{11}(A)M - Ma_{22}(A)$ and $d_{21}(NA) = d_{21}(N)A + Na_{11}(A) - a_{22}(A)N$. Taking $A = I_{\mathcal{A}}$ leads to $a_{11}(I_{\mathcal{A}})M = Ma_{22}(I_{\mathcal{A}})$ and $Na_{11}(I_{\mathcal{A}}) = a_{22}(I_{\mathcal{A}})N$. So $MNa_{11}(I_{\mathcal{A}}) = Ma_{22}(I_{\mathcal{A}})N = a_{11}(I_{\mathcal{A}})MN$ and hence (2.20) can be abbreviated to

$$a_{11}(MN) = c_{12}(M)N + Md_{21}(N) + b_{11}(NM). \tag{2.22}$$

By (2.8)-(2.11), (2.14)-(2.19), (2.21), and (2.22), the proof is complete.

□

The center $Z(\mathcal{U})$ of $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$ is of the form given in the proposition below, and since the proof is analogous to that of [3, Proposition 3], we omit it.

Proposition 2.3. *The center of* $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$ *is*

$$Z(\mathcal{U}) = \{A \oplus B : A \in Z(\mathcal{A}),\ B \in Z(\mathcal{B}),\ AM = MB,$$
$$NA = BN,\ \forall M \in \mathcal{M},\ \forall N \in \mathcal{N}\}.$$

Furthermore, if one of the following conditions holds:

(1) *M is a faithful left* $\mathcal{A}$*-module and a faithful right* $\mathcal{B}$*-module*;

(2) *M is a faithful left* $\mathcal{A}$*-module and N is a faithful left* $\mathcal{B}$*-module*;

(3) *M is a faithful right* $\mathcal{B}$*-module and N is a faithful right* $\mathcal{A}$*-module*;

(4) *N is a faithful left* $\mathcal{B}$*-module and a faithful right* $\mathcal{A}$*-module.*

Then there exists a unique isomorphism τ *from* $\pi_{\mathcal{B}}(Z(\mathcal{U}))$ *to* $\pi_{\mathcal{A}}(Z(\mathcal{U}))$ *such that* $\tau(B)M = MB$ *and* $N\tau(B) = BN$ *for any* $B \in \mathcal{B}$, $M \in \mathcal{M}$, *and* $N \in \mathcal{N}$.

The proof of the proposition below concerning the structure of derivations on $\mathcal{U}$ is standard and we omit it.

Proposition 2.4. *A linear mapping* δ *on* $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$ *is a derivation*, *if and only if it is of the form*

$$\delta\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right) = \begin{bmatrix} a_{11}(A) - MN_0 - M_0N & AM_0 - M_0B + c_{12}(M) \\ N_0A - BN_0 + d_{21}(N) & b_{22}(B) + N_0M + NM_0 \end{bmatrix},$$

where $N_0 \in \mathcal{N}$, $M_0 \in \mathcal{M}$, *and* $a_{11} : \mathcal{A} \to \mathcal{A}, b_{22} : \mathcal{B} \to \mathcal{B}, c_{12} : \mathcal{M} \to \mathcal{M}$, $d_{21} : \mathcal{N} \to \mathcal{N}$ *are linear mappings satisfying*

(1) a_{11} *is a derivation on* $\mathcal{A}$, $c_{12}(AM) = a_{11}(A)M + Ac_{12}(M)$ *and* $d_{21}(NA) = Na_{11}(A) + d_{21}(N)A$;

(2) b_{22} *is a derivation on* $\mathcal{B}$, $c_{12}(MB) = Mb_{22}(B) + c_{12}(M)B$ *and* $d_{21}(BN) = b_{22}(B)N + Bd_{21}(N)$;

(3) $a_{11}(MN) = c_{12}(M)N + Md_{21}(N)$ *and* $b_{22}(NM) = d_{21}(N)M + Nc_{12}(M)$.

Now, we are in a position to prove our main theorem.

Proof of Theorem 2.1. From Lemma 2.2, it follows that for any $A \in \mathcal{A}$, $M \in \mathcal{M}$, $N \in \mathcal{N}$, and $B \in \mathcal{B}$:

$$L\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right)$$

$$= \begin{bmatrix} a_{11}(A) - MN_0 - M_0N + b_{11}(B) & AM_0 + c_{12}(M) - M_0B \\ N_0A + d_{21}(N) - BN_0 & a_{22}(A) + N_0M + NM_0 + b_{22}(B) \end{bmatrix},$$

where $M_0 \in M$ and $N_0 \in N$.

We assume that (1) holds. The proofs for the other cases are analogous.

By Lemma 2.2, $a_{22}(A) \in Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{U}))$, since M is both a faithful left $\mathcal{A}$-module and a faithful right $\mathcal{B}$-module, by Proposition 2.3, there exists a unique isomorphism $\tau : \pi_{\mathcal{B}}(Z(\mathcal{U})) \to \pi_{\mathcal{A}}(Z(\mathcal{U}))$ such that

$$\tau(a_{22}(A))M = Ma_{22}(A) \quad \text{and} \quad N\tau(a_{22}(A)) = a_{22}(A)N.$$

Similarly, since $b_{11}(B) \in Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{U}))$, we have

$$\tau^{-1}(b_{11}(B))N = Nb_{11}(B) \quad \text{and} \quad b_{11}(B)M = M\tau^{-1}(b_{11}(B)).$$

Let $l_{\mathcal{A}} = \tau \circ a_{22}$ and $l_{\mathcal{B}} = \tau^{-1} \circ b_{11}$. We have $l_{\mathcal{A}}(A)M = Ma_{22}(A)$, $Nl_{\mathcal{A}}(A) = a_{22}(A)N$, $l_{\mathcal{B}}(B)N = Nb_{11}(B)$, and $b_{11}(B)M = Ml_{\mathcal{B}}(B)$. Let $a'_{11} = a_{11} - l_{\mathcal{A}}$ and $b'_{22} = b_{22} - l_{\mathcal{B}}$. Then

$$L\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right) = \begin{bmatrix} a'_{11}(A) - MN_0 - M_0 N & AM_0 + c_{12}(M) - M_0 B \\ N_0 A + d_{21}(N) - BN_0 & N_0 M + NM_0 + b'_{22}(B) \end{bmatrix},$$

$$+ \begin{bmatrix} l_{\mathcal{A}}(A) + b_{11}(B) & 0 \\ 0 & a_{22}(A) + l_{\mathcal{B}}(B) \end{bmatrix}.$$

Let

$$\delta\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right) = \begin{bmatrix} a'_{11}(A) - MN_0 - M_0 N & AM_0 + c_{12}(M) - M_0 B \\ N_0 A + d_{21}(N) - BN_0 & N_0 M + NM_0 + b'_{22}(B) \end{bmatrix},$$

and

$$h\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right) = \begin{bmatrix} l_{\mathcal{A}}(A) + b_{11}(B) & 0 \\ 0 & a_{22}(A) + l_{\mathcal{B}}(B) \end{bmatrix}.$$

We claim that δ is a derivation on $\mathcal{U}$ and $h : \mathcal{U} \to Z(\mathcal{U})$ is a linear map, vanishing at commutators $[S, T]$ with $ST = 0$.

Claim 1. By Lemma 2.1, for any $A \in \mathcal{A}$ and $M \in \mathcal{M}$,

$$\begin{aligned} c_{12}(AM) &= Ac_{12}(M) + a_{11}(A)M - Ma_{22}(A) \\ &= Ac_{12}(M) + a_{11}(A)M - l_{\mathcal{A}}(A)M \\ &= Ac_{12}(M) + a'_{11}(A)M, \\ d_{21}(NA) &= d_{21}(N)A + Na_{11}(A) - a_{22}(A)N \\ &= d_{21}(N)A + Na_{11}(A) - Nl_{\mathcal{A}}(A) \\ &= d_{21}(N)A + Na'_{11}(A). \end{aligned}$$

So for any $A_1, A_2 \in \mathcal{A}$ and $M \in \mathcal{M}$,

$$\begin{aligned} c_{12}(A_1 A_2 M) &= A_1 A_2 c_{12}(M) + a'_{11}(A_1 A_2)M, \\ c_{12}(A_1 A_2 M) &= a'_{11}(A_1)A_2 M + A_1 c_{12}(A_2 M) \\ &= a'_{11}(A_1)A_2 M + A_1 A_2 c_{12}(M) + A_1 a'_{11}(A_2)M. \end{aligned}$$

So $(a'_{11}(A_1A_2) - a'_{11}(A_1)A_2 - A_1a'_{11}(A_2))M = 0$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, we have a'_{11} is a derivation. Similarly, one can show that $c_{12}(MB) = c_{12}(M)B + Mb'_{22}(B)$, $d_{21}(BN) = b'_{22}(B)N + Bd_{21}(N)$, and b'_{22} is a derivation.

Since $MN \in \mathcal{A}$ and $NM \in \mathcal{B}$, we have

$$\begin{aligned} c_{12}(MNM) &= MNc_{12}(M) + a_{11}(MN)M - Ma_{22}(MN) \\ &= MNc_{12}(M) + c_{12}(M)NM + Md_{21}(N)M \\ &\quad + b_{11}(NM)M - Ma_{22}(MN), \\ c_{12}(MNM) &= c_{12}(M)NM + Mb_{22}(NM) - b_{11}(NM)M \\ &= c_{12}(M)NM + MNc_{12}(M) + Md_{21}(N)M \\ &\quad + Ma_{22}(MN) - b_{11}(NM)M. \end{aligned}$$

So $Ma_{22}(MN) = b_{11}(NM)M$. On the other hand, $Ma_{22}(MN) = l_{\mathcal{A}}(MN)M$. Hence $b_{11}(NM) = l_{\mathcal{A}}(MN)$ and $a'_{11}(MN) = c_{12}(M)N + Md_{21}(N)$. Similarly, one can show that $b'_{22}(NM) = Nc_{12}(M) + d_{21}(N)M$. By Proposition 2.4, δ is a derivation.

Claim 2. It is easy to show that h is a linear mapping with its image in $Z(\mathcal{U})$. Then for $S, T \in \mathcal{U}$ with $ST = 0$, we have

$$\begin{aligned} \delta([S, T]) + h([S, T]) = L([S, T]) &= [L(S), T] + [S, L(T)] \\ &= [\delta(S), T] + [S, \delta(T)] + [h(S), T] + [S, h(T)] \\ &= [\delta(S), T] + [S, \delta(T)]. \end{aligned}$$

Hence $h([S, T]) = 0$. This concludes the proof. □

Obviously, when $N = 0$, $\mathcal{U}$ degenerates to an upper triangular algebra. Thus, we have the following corollary:

Corollary 2.5. *Let* $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *be a triangular algebra, where* $\mathcal{M}$ *be a faithful left* $\mathcal{A}$*-module and a faithful right* $\mathcal{B}$*-module. If L is a Lie derivable mapping at* 0 *from* $\mathcal{U}$ *into itself and* $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{U}))$, $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{U}))$, *then* L *can be expressed as* $\delta + h$, *where* δ *is a derivation on* $\mathcal{U}$ *and* $h : \mathcal{U} \to Z(\mathcal{U})$ *is a linear mapping, vanishing at commutators* $[S, T]$ *with* $ST = 0$.

3. Lie Derivable Mapping at $I_{\mathcal{A}} \oplus 0$-Product Elements

In this section, we study the Lie derivable mapping at $I_{\mathcal{A}} \oplus 0$-product elements.

Theorem 3.1. *Let* $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$ *be a generalized matrix algebra. Suppose that for every* $A \in \mathcal{A}$, *there exists an integer* n *such that* $nI_{\mathcal{A}} - A$ *is invertible in* $\mathcal{A}$. *If L is a Lie derivable mapping at* $I_{\mathcal{A}} \oplus 0$ *from* $\mathcal{U}$ *into itself,* $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{U}))$, $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{U}))$, *and one of the following conditions holds*:

(1) *M is a faithful left* $\mathcal{A}$*-module and a faithful right* $\mathcal{B}$*-module*;

(2) *M is a faithful left* $\mathcal{A}$*-module and N is a faithful left* $\mathcal{B}$*-module*;

(3) *M is a faithful right* $\mathcal{B}$*-module and N is a faithful right* $\mathcal{A}$*-module*;

(4) *N is a faithful left* $\mathcal{B}$*-module and a faithful right* $\mathcal{A}$*-module.*

Then L can be expressed as $\delta + h$, *where* δ *is a derivation on* $\mathcal{U}$, *and* $h : \mathcal{U} \to Z(\mathcal{U})$ *is a linear mapping, vanishing at commutators* $[S, T]$, *where* $S, T \in \mathcal{U}$ *with* $ST = I_{\mathcal{A}} \oplus 0$.

To prove Theorem 3.1, we first show a lemma.

Lemma 3.2. *Let* $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$ *be a generalized matrix algebra. Suppose that for every* $A \in \mathcal{A}$, *there exists an integer* n *such that* $nI_{\mathcal{A}} - A$ *is invertible in* $\mathcal{A}$. *If L is a Lie derivable mapping at* $I_{\mathcal{A}} \oplus 0$ *from* $\mathcal{U}$ *into itself, then*

$$L\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right)$$

$$= \begin{bmatrix} a_{11}(A) - MN_0 - M_0N + b_{11}(B) & AM_0 + c_{12}(M) - M_0B \\ N_0A + d_{21}(N) - BN_0 & a_{22}(A) + N_0M + NM_0 + b_{22}(B) \end{bmatrix},$$

where $M_0 \in M$, $N_0 \in N$, *and* $a_{11} : \mathcal{A} \to \mathcal{A}, b_{22} : \mathcal{B} \to \mathcal{B}, a_{22} : \mathcal{A} \to Z(\mathcal{B})$, $b_{11} : \mathcal{B} \to Z(\mathcal{A})$ *are all linear mappings satisfying*:

$$c_{12}(AM) = Ac_{12}(M) + a_{11}(A)M - Ma_{22}(A),$$

$$c_{12}(MB) = c_{12}(M)B + Mb_{22}(B) - b_{11}(B)M,$$

$$d_{21}(NA) = d_{21}(N)A + Na_{11}(A) - a_{22}(A)N,$$

$$d_{21}(BN) = Bd_{21}(N) + b_{22}(B)N - Nb_{11}(B),$$

$$a_{11}(MN) = c_{12}(M)N + Md_{21} + b_{11}(NM),$$

$$b_{22}(NM) = Nc_{12}(M) + d_{21}(N)M + a_{22}(MN).$$

Proof. We prove the lemma by several steps.

Step 1. For any $A_1, A_2 \in \mathcal{A}$ with $A_1A_2 = I_{\mathcal{A}}$ and $B_1, B_2 \in \mathcal{B}$ with $B_1B_2 = 0$, let $S = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}$ and $T = \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}$. Then $ST = I_{\mathcal{A}} \oplus 0$ and

$$\begin{bmatrix} a_{11}([A_1, A_2]) + b_{11}([B_1, B_2]) & a_{12}([A_1, A_2]) + b_{12}([B_1, B_2]) \\ a_{21}([A_1, A_2]) + b_{21}([B_1, B_2]) & a_{22}([A_1, A_2]) + b_{22}([B_1, B_2]) \end{bmatrix}$$

$$= L([S, T]) = [L(S), T] + [S, L(T)]$$

$$= \begin{bmatrix} a_{11}(A_1) + b_{11}(B_1) & a_{12}(A_1) + b_{12}(B_1) \\ a_{21}(A_1) + b_{21}(B_1) & a_{22}(A_1) + b_{22}(B_1) \end{bmatrix}\begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}$$

$$- \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}\begin{bmatrix} a_{11}(A_1) + b_{11}(B_1) & a_{12}(A_1) + b_{12}(B_1) \\ a_{21}(A_1) + b_{21}(B_1) & a_{22}(A_1) + b_{22}(B_1) \end{bmatrix}$$

$$+ \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}\begin{bmatrix} a_{11}(A_2) + b_{11}(B_2) & a_{12}(A_2) + b_{12}(B_2) \\ a_{21}(A_2) + b_{21}(B_2) & a_{22}(A_2) + b_{22}(B_2) \end{bmatrix}$$

$$-\begin{bmatrix} a_{11}(A_2) + b_{11}(B_2) & a_{12}(A_2) + b_{12}(B_2) \\ a_{21}(A_2) + b_{21}(B_2) & a_{22}(A_2) + b_{22}(B_2) \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}.$$

The above matrix equation implies the following four equations:

$$a_{11}([A_1, A_2]) + b_{11}([B_1, B_2])$$
$$= [a_{11}(A_1), A_2] + [b_{11}(B_1), A_2] + [A_1, a_{11}(A_2)] + [A_1, b_{11}(B_2)], \tag{3.1}$$

$$a_{12}([A_1, A_2]) + b_{12}([B_1, B_2])$$
$$= a_{12}(A_1)B_2 + b_{12}(B_1)B_2 - A_2 a_{12}(A_1) - A_2 b_{12}(B_1)$$
$$+ A_1 a_{12}(A_2) + A_1 b_{12}(B_2) - a_{12}(A_2)B_1 - b_{12}(B_2)B_1, \tag{3.2}$$

$$a_{21}([A_1, A_2]) + b_{21}([B_1, B_2])$$
$$= a_{21}(A_1)A_2 + b_{21}(B_1)A_2 - B_2 a_{21}(A_1) - B_2 b_{21}(B_1)$$
$$+ B_1 a_{21}(A_2) + B_1 b_{21}(B_2) - a_{21}(A_2)A_1 - b_{21}(B_2)A_1, \tag{3.3}$$

$$a_{22}([A_1, A_2]) + b_{22}([B_1, B_2])$$
$$= [a_{22}(A_1), B_2] + [b_{22}(B_1), B_2] + [B_1, a_{22}(A_2)] + [B_1, b_{22}(B_2)]. \tag{3.4}$$

Taking $B_1 = B_2 = 0$ in (3.1)-(3.4), we obtain

$$a_{11}([A_1, A_2]) = [a_{11}(A_1), A_2] + [A_1, a_{11}(A_2)],$$

$$a_{12}([A_1, A_2]) = A_1 a_{12}(A_2) - A_2 a_{12}(A_1),$$

$$a_{21}([A_1, A_2]) = a_{21}(A_1)A_2 - a_{21}(A_2)A_1,$$

$$a_{22}([A_1, A_2]) = 0,$$

for any $A_1, A_2 \in \mathcal{A}$ with $A_1 A_2 = I_{\mathcal{A}}$. Hence, by taking $B_2 = 0$ in (3.1)-(3.4), we have

$$b_{11}(B)A_2 = A_2 b_{11}(B) \quad \text{and} \quad B a_{22}(A_2) = a_{22}(A_2)B,$$

$$A_2 b_{12}(B) = -a_{12}(A_2)B \quad \text{and} \quad b_{21}(B)A_2 = -B a_{21}(A_2),$$

for any $B \in \mathcal{B}$. Note that the above equations are true for all invertible elements in $\mathcal{A}$. Since for every $A \in \mathcal{A}$, there exists an integer n such that $nI_{\mathcal{A}} - A$ is invertible in $\mathcal{A}$, we have

$$b_{11}(B)A = Ab_{11}(B) \quad \text{and} \quad Ba_{22}(A) = a_{22}(A)B, \tag{3.5}$$

$$Ab_{12}(B) = -a_{12}(A)B \quad \text{and} \quad b_{21}(B)A = -Ba_{21}(A), \tag{3.6}$$

for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

By (3.5), we have $b_{11}(B) \in Z(\mathcal{A})$ for any $B \in \mathcal{B}$ and $a_{22}(A) \in Z(\mathcal{B})$ for any $A \in \mathcal{A}$.

By (3.6), let $M_0 = a_{12}(I_{\mathcal{A}})$ and $N_0 = a_{21}(I_{\mathcal{A}})$, we have

$$a_{12}(A) = AM_0, \quad b_{12}(B) = -M_0B, \quad a_{21}(A) = N_0A$$

and

$$b_{21}(B) = -BN_0, \tag{3.7}$$

for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Step 2. For any $A_1, A_2 \in \mathcal{A}$ with $A_1A_2 = I_{\mathcal{A}}$ and $M \in \mathcal{M}$, let $S = \begin{bmatrix} A_1 & M \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix}$. Then $ST = I_{\mathcal{A}} \oplus 0$ and

$$\begin{bmatrix} a_{11}([A_1, A_2]) - c_{11}(A_2M) & a_{12}([A_1, A_2]) - c_{12}(A_2M) \\ a_{21}([A_1, A_2]) - c_{21}(A_2M) & -c_{22}(A_2M) \end{bmatrix}$$

$$= L([S, T]) = [L(S), T] + [S, L(T)]$$

$$= \begin{bmatrix} a_{11}(A_1) + c_{11}(M) & a_{12}(A_1) + c_{12}(M) \\ a_{21}(A_1) + c_{21}(M) & a_{22}(A_1) + c_{22}(M) \end{bmatrix} \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11}(A_1) + c_{11}(M) & a_{12}(A_1) + c_{12}(M) \\ a_{21}(A_1) + c_{21}(M) & a_{22}(A_1) + c_{22}(M) \end{bmatrix}$$

$$+ \begin{bmatrix} A_1 & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11}(A_2) & a_{12}(A_2) \\ a_{21}(A_2) & a_{22}(A_2) \end{bmatrix}$$

$$-\begin{bmatrix} a_{11}(A_2) & a_{12}(A_2) \\ a_{21}(A_2) & a_{22}(A_2) \end{bmatrix}\begin{bmatrix} A_1 & M \\ 0 & 0 \end{bmatrix}.$$

The above matrix equation implies the following four equations:

$$c_{11}(A_2M) = [A_2, c_{11}(M)] - Ma_{21}(A_2), \tag{3.8}$$

$$c_{12}(A_2M) = A_2c_{12}(M) - Ma_{22}(A_2) + a_{11}(A_2)M, \tag{3.9}$$

$$c_{21}(A_2M) = -c_{21}(M)A_2, \tag{3.10}$$

$$c_{22}(A_2M) = a_{21}(A_2)M, \tag{3.11}$$

for any $M \in \mathcal{M}$. By choosing $A_2 = I_{\mathcal{A}}$ in (3.8), (3.10), and (3.11), we have

$$c_{11}(M) = -MN_0, \quad c_{21}(M) = 0, \text{ and } c_{22}(M) = N_0M, \tag{3.12}$$

for any $M \in \mathcal{M}$. Note that the Equation (3.9) is true for any invertible element in $\mathcal{A}$. Since for every $A \in \mathcal{A}$, there exists an integer n such that $nI_{\mathcal{A}} - A$ is invertible, we have that

$$c_{12}(AM) = Ac_{12}(M) - Ma_{22}(A) + a_{11}(A)M, \tag{3.13}$$

for any $A \in \mathcal{A}$ and $M \in \mathcal{M}$.

Symmetrically, by considering $S = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} A_2 & 0 \\ N & 0 \end{bmatrix}$ with $A_1A_2 = I_{\mathcal{A}}$, we arrive at

$$d_{11}(N) = -M_0N, \quad d_{12}(N) = 0, \quad \text{and} \quad d_{22}(N) = NM_0, \tag{3.14}$$

$$d_{21}(NA) = d_{21}(N)A - a_{22}(A)N + Na_{11}(A), \tag{3.15}$$

for any $A \in \mathcal{A}$ and $N \in \mathcal{N}$.

Step 3. For any $M \in \mathcal{M}$ and $B \in \mathcal{B}$, let $S = \begin{bmatrix} I_{\mathcal{A}} & -M \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} I_{\mathcal{A}} & MB \\ 0 & B \end{bmatrix}$. Then $ST = I_{\mathcal{A}} \oplus 0$ and

$$\begin{bmatrix} * & c_{12}(M) \\ * & * \end{bmatrix} = L([S, T]) = [L(S), T] + [S, L(T)]$$

$$= \begin{bmatrix} * & \begin{matrix} a_{11}(I_{\mathcal{A}})MB - c_{12}(M)B + c_{12}(M) - MBa_{22}(I_{\mathcal{A}}) \\ + c_{12}(MB) - Ma_{22}(I_{\mathcal{A}}) - Mb_{22}(B) + a_{11}(I_{\mathcal{A}})M + b_{11}(B)M \end{matrix} \\ * & * \end{bmatrix},$$

where $*$ denotes the omitted matrix element.

It follows that

$$c_{12}(MB) = c_{12}(M)B - b_{11}(B)M + Mb_{22}(B) + MBa_{22}(I_{\mathcal{A}})$$
$$- a_{11}(I_{\mathcal{A}})MB + Ma_{22}(I_{\mathcal{A}}) - a_{11}(I_{\mathcal{A}})M,$$

for every $B \in \mathcal{B}$ and every $M \in \mathcal{M}$. Taking $A = I_{\mathcal{A}}$ in (3.13) gives $Ma_{22}(I_{\mathcal{A}}) = a_{11}(I_{\mathcal{A}})M$. Hence

$$c_{12}(MB) = c_{12}(M)B - b_{11}(B)M + Mb_{22}(B), \tag{3.16}$$

for every $B \in \mathcal{B}$ and every $M \in \mathcal{M}$.

Symmetrically, by considering $S = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ BN & B \end{bmatrix}$ and $T = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ -N & 0 \end{bmatrix}$ with $A_1A_2 = I_{\mathcal{A}}$, we arrive at

$$d_{21}(BN) = Bd_{21}(N) + b_{22}(B)N - Nb_{11}(B), \tag{3.17}$$

for every $B \in \mathcal{A}$ and every $N \in \mathcal{N}$.

Step 4. For any $M \in \mathcal{M}$ and $N \in \mathcal{N}$, let $S = \begin{bmatrix} I_{\mathcal{A}} + MN & -M \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ N & 0 \end{bmatrix}$. Then $ST = I_{\mathcal{A}} \oplus 0$ and

$$- \begin{bmatrix} \begin{matrix} a_{11}(MN) + MN_0 - M_0N \\ - M_0NMN - b_{11}(NM) \end{matrix} & MNM_0 - c_{12}(M) + M_0NM \\ N_0MN + d_{21}(N + NMN) + NMN_0 & \begin{matrix} a_{22}(MN) - N_0M + NM_0 \\ + NMNM_0 - b_{22}(NM) \end{matrix} \end{bmatrix}$$

$$= L([S, T]) = [L(S), T] + [S, L(T)]$$

$$= \begin{bmatrix} a_{11}(I_{\mathcal{A}} + MN) + MN_0 & M_0 + MNM_0 - c_{12}(M) \\ N_0 + N_0MN & a_{22}(I_{\mathcal{A}} + MN) - N_0M \end{bmatrix} \begin{bmatrix} I_{\mathcal{A}} & 0 \\ N & 0 \end{bmatrix}$$

$$- \begin{bmatrix} I_{\mathcal{A}} & 0 \\ N & 0 \end{bmatrix} \begin{bmatrix} a_{11}(I_{\mathcal{A}} + MN) + MN_0 & M_0 + MNM_0 - c_{12}(M) \\ N_0 + N_0MN & a_{22}(I_{\mathcal{A}} + MN) - N_0M \end{bmatrix}$$

$$+ \begin{bmatrix} I_{\mathcal{A}} + MN & -M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11}(I_{\mathcal{A}}) - M_0N & M_0 \\ N_0 + d_{21}(N) & a_{22}(I_{\mathcal{A}}) + NM_0 \end{bmatrix}$$

$$- \begin{bmatrix} a_{11}(I_{\mathcal{A}}) - M_0N & M_0 \\ N_0 + d_{21}(N) & a_{22}(I_{\mathcal{A}}) + NM_0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{A}} + MN & -M \\ 0 & 0 \end{bmatrix}.$$

The above matrix relation implies

$$a_{11}(MN) = c_{12}(M)N + Md_{21}(N) + b_{11}(NM)$$

$$+ a_{11}(I_{\mathcal{A}})MN - MNa_{11}(I_{\mathcal{A}}), \tag{3.18}$$

$$b_{22}(NM) = Nc_{12}(M) + d_{21}(N)M + a_{22}(MN). \tag{3.19}$$

Taking $A = I_{\mathcal{A}}$ in (3.15) leads to $Na_{11}(I_{\mathcal{A}}) = a_{22}(I_{\mathcal{A}})N$. So $MNa_{11}(I_{\mathcal{A}}) = Ma_{22}(I_{\mathcal{A}})N = a_{11}(I_{\mathcal{A}})MN$ and hence (3.18) can be abbreviated to

$$a_{11}(MN) = c_{12}(M)N + Md_{21}(N) + b_{11}(NM). \tag{3.20}$$

By (3.5), (3.7), and (3.12)-(3.20), the proof is complete. □

Proof of Theorem 3.1. Substitute Lemma 2.2 by Lemma 3.2 in Theorem 2.1, one can show that Theorem 3.1 is true and we leave it to the readers. □

Corollary 3.3. *Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra and $\mathcal{M}$ be a faithful left $\mathcal{A}$-module and a faithful right $\mathcal{B}$-module. Suppose that for every element $A \in \mathcal{A}$, there exists an integer n such that $nI_{\mathcal{A}} - A$ is invertible in $\mathcal{A}$. If L is a Lie derivable mapping at $I_{\mathcal{A}} \oplus 0$ from $\mathcal{U}$ into itself, $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathcal{U}))$, $Z(\mathcal{B}) = \pi_{\mathcal{B}}(Z(\mathcal{U}))$, then L can be expressed as*

$\delta + h$, *where* δ *is a derivation on* $\mathcal{U}$ *and* $h : \mathcal{U} \to Z(\mathcal{U})$ *is a linear mapping, vanishing at commutators* $[S, T]$ *with* $ST = I_{\mathcal{A}} \oplus 0$.

Acknowledgement

This work is supported by NSF of China.

References

[1] J. Alaminos, M. Mathieu and A. Villena, Symmetric amenability and Lie derivations, Math. Proc. Cambridge Philos. Soc. 137 (2004), 433-439.

[2] M. Brešar, Characterizing homomorphisms, derivations and multipliers in rings with idempotents, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007), 9-21.

[3] W. Cheung, Lie derivations of triangular algebra, Linear Multilinear Algebra 51 (2003), 299-310.

[4] B. Johnson, Symmetric amenability and the nonexistence of Lie and Jordan derivations, Math. Proc. Cambridge Philos. Soc. 120 (1996), 455-473.

[5] J. Li and Z. Pan, Annihilator-preserving maps, multipliers, and derivations, Linear Algebra Appl. 432 (2010), 5-13.

[6] F. Lu, Lie derivation of certain CSL algebras, Israel J. Math. 155 (2006), 149-156.

[7] F. Lu and B. Liu, Lie derivations of reflexive algebras, Integr. Equ. Oper. Theory 64 (2009), 261-271.

[8] F. Lu and W. Jing, Characterizations of Lie derivation of $B(X)$, Linear Algebra Appl. 432 (2010), 89-99.

[9] M. Mathieu and A. Villena, The structure of Lie derivations on C^*-algebras, J. Funct. Anal. 202 (2003), 504-525.

[10] C. Miers, Lie derivations of von Neumann algebras, Duke Math. J. 40 (1973), 403-409.

[11] A. Sands, Radicals and Morita contexts, J. Algebra 24 (1973), 335-345.

[12] L. Zhang, J. Zhu and J. Wu, All-derivable points in nest algebras, Linear Algebra Appl. 433 (2010), 91-100.

■